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The spectral geometry of the Hopf fibration

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Abstract. We study the spectral geometry of the Hopf fibration $S^1 \to S^3 \to S^2$ and determine the right invariant metrics on S^3 for which there exist eigenforms of the Laplacian on S^2 which pull back to eigenforms of the Laplacian on S^3. We show that the pull-back of the volume form on S^2 can be an eigenform of the Laplacian on S^3 with non-zero eigenvalue. We show that if $G \to P \to Y$ is a principal bundle with a bundle metric and that if $H^1(G; \mathbb{C}) = 0$, then eigenvalues cannot change. Thus eigenvalues do not change for the fibrations $SO(n-1) \to SO(n) \to S^{n-1}$ and $SPIN(n-1) \to SPIN(n) \to S^{n-1}$ if $n \geqslant 4$. We also study the corresponding questions in the complex category for the fibration of the Hopf manifold $S^1 \times S^3 \to S^2$.

1. Introduction

Bérard Bergery and Bourguignon [4] discuss the Laplacian of a Riemannian submersion and provide an application to quantum physics. They note: ‘Recently, there has been a renewed interest in classical physics for non-bijective canonical transformations (see [5, 18]). This very general expression should not be taken literally, but more in the sense that certain interesting maps between configuration spaces turn out to be nonlinear and non-bijective. From a mathematician’s point of view, these maps are in fact extremely nice (namely, coverings or Hopf fibrations in the examples that we detail later). When going to the quantum level, one has to describe how the spectrum of the quantum operators are related. Once more, the quantum operators are not the most general operators, but very natural ones related to the Riemannian geometry of the situation (for example the Laplace operator of a Riemannian metric plus a potential for the energy).’

Boiteux [5] studies the Coulumb potential in two and three dimensions and shows that non-bijective transformations require a fibre-bundle formulation of mechanics. He shows that the Hopf fibration leads to an inverse harmonic oscillator problem. Boiteux notes ‘in quantum mechanics, those transformations connect operators with different spectra which as such cannot be deduced from one another by unitary transformations’. Kibler and Négada [14, 15] extend the work of Boiteux to discuss the Kustaanheimo–Stiefel transformation for the hydrogen atom and to discuss the Stark and Zeeman effects in hydrogen ions. For other related work on non-bijective canonical transformations, we refer to Asorey *et al* [1],

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Cerdeira [7], Dehghani and Sobouti [8], Gracia-Bondía [13], Kibler *et al* [16] and to Kibler and Winternitz [17].

The present paper is devoted to the study of the spectral geometry of the Hopf fibration and of certain generalizations; the Hopf fibration arises as a regularization of Kepler motion [4]. We shall be interested in the relationship between the spectrum of the base and of the total space of the bundle provided by pull-back; this is the transformation discussed by Boiteux [5].

We establish some notational conventions. If M is a closed Riemannian manifold, let Δ_M^p be the Laplace Beltrami operator on the space of smooth p forms $C^\infty\Lambda^p M$. Let

$$E(\lambda, \Delta_M^p) = \{\Phi_p \in C^\infty\Lambda^p M : \Delta_M^p\Phi_p = \lambda\Phi_p\}$$

be the eigenspaces of Δ_M^p. Let $\pi : Z \to Y$ be a Riemannian submersion where Z and Y are closed Riemannian manifolds. Pull back defines a natural map

$$\pi^* : C^\infty\Lambda^p Y \to C^\infty\Lambda^p Z.$$

If the pull-back of every eigenform of the Laplacian on Y is an eigenform of the Laplacian on Z, then the eigenvalue cannot change, see theorem 2.1 for details. In this paper, we will be interested in the case in which a single eigenvalue can change. That means we want to construct examples of a p form Φ_p so that $\Phi_p \in E(\lambda, \Delta_Y^p)$ and $\pi^*\Phi_p \in E(\mu, \Delta_Y^p)$ for $\lambda \neq \mu$. It is known this is not possible if $p = 0$ or if $\mu < \lambda$, see theorem 2.1 for details. Muto [19, 20] constructed examples with $p = 2$ where eigenvalues can change. The simplest example is provided by the Hopf fibration $\pi : (S^3, g_3) \to (S^2, \frac{1}{4}g_2)$ where g_n is the standard metric on S^n. If ν_2 is the volume form on S^2, then $\nu_2 \in E(0, \Delta_{S^2}^2)$ and $\pi^*\nu_2 \in E(4, \Delta_{S^3}^2)$.

In section 2, we study the real spectral geometry of the Hopf fibration. We identify S^3 with the unit quaternions or equivalently with $SU(2)$ to see that S^3 is a Lie group. Let ζ^1, ζ^2, ζ^3 be the usual orthonormal basis for the set of right invariant 1-forms where ζ^1 spans the vertical space of π; see section 2 for more precise definitions. If $\tilde{g}_3$ is a right invariant metric on S^3 so that $\pi : (S^3, \tilde{g}_3) \to (S^2, \frac{1}{4}g_2)$ is a Riemannian submersion, then there exist $\alpha > 0$ and $\boldsymbol{\beta} = (\beta_2, \beta_3)$ so that

$$\tilde{g}_3 := \alpha^2(\zeta^1 - \beta_2\zeta^2 - \beta_3\zeta^3) \circ (\zeta^1 - \beta_2\zeta^2 - \beta_3\zeta^3) + \zeta^2 \circ \zeta^2 + \zeta^3 \circ \zeta^3.$$

The fibre circles of π are geodesics in the metric $\tilde{g}_3$ if and only if $\boldsymbol{\beta} = 0$. If $\boldsymbol{\beta} = 0$, we show that $\pi^*\nu_2 \in E(4\alpha^2, \Delta_{S^3}^2)$ and that no other eigenvalue changes. If $\boldsymbol{\beta} \neq 0$, we show no eigenvalue changes; see theorem 2.3 for details.

In section 3, we study the principal bundles. Let G be a compact Lie group with vanishing first de Rham cohomology group. Let $G \to P \xrightarrow{\pi} Y$ be a principal bundle with fibre G. If we choose a bundle metric for P, we show eigenvalues cannot change; thus in particular eigenvalues cannot change for the fibrations $SO(n-1) \to SO(n) \to S^{n-1}$ or $SPIN(n-1) \to SPIN(n) \to S^{n-1}$ if $n \geqslant 4$. We note that the Hopf fibration is the fibration $SPIN(2) \to SPIN(3) \to S^2$. Conversely, if $H^1(G; \mathbb{C}) \neq 0$, we show there exists a principal G bundle over S^3 where an eigenvalue changes.

In section 4, we generalize the Hopf fibration to the complex setting and study the holomorphic fibration $\pi : S^1 \times S^3 \to S^2$ of the Hopf manifold. We will show that $\nu_2 \in E(0, \Delta_{S^2}^{1,1})$ and $\pi^*\nu_2 \in E(\tilde{\mu}, \Delta_{S^1\times S^3}^{1,1})$ for $\tilde{\mu} > 0$ so again this provides an example where the eigenvalue of the complex Laplacian on forms of degree $(1, 1)$ can change. We discuss the spectral resolution of the complex Laplacian $\Delta^{0,0}$ in terms of the real Laplacian for the Hopf manifold; this is of interest as the Hopf manifold is not Kaehler.

The spectral geometry of principal bundles is an important one in gauge-field theory and we hope these results will be useful in further investigations.

2. The real spectral geometry of the Hopf fibration

We begin by reviewing the geometry of Riemannian submersions. Let $\pi : Z \to Y$ be a Riemannian submersion. We shall use capital letters for tensors and forms on the base manifold Y and lower-case letters for tensors and forms on the total space Z. Let $\rho_{\mathcal{V}}$ and $\rho_{\mathcal{H}}$ be orthogonal projection on the vertical and horizontal distributions $\mathcal{V}$ and $\mathcal{H}$ of the submersion. Let $\{e_i\}$ be a local orthonormal frame for $\mathcal{V}$ and let $\{f_a\}$ be a local orthonormal frame for $\mathcal{H}$. Let ∇^Z be the Levi-Civita connection on Z. Define the non-normalized mean curvature vector θ and the curvature tensor ω by

$$\theta := \rho_{\mathcal{H}}(\nabla^Z_{e_i} e_i) \qquad \text{and} \qquad \omega_{abi} = \tfrac{1}{2} g([f_a, f_b], e_i).$$

The fibres are minimal if and only if $\theta = 0$; the horizontal distribution is integrable if and only if $\omega = 0$. We let ω act on the exterior algebra by defining

$$\mathcal{E} := \omega_{abi}\, \mathrm{ext}(e^i)\, \mathrm{int}(f^a)\, \mathrm{int}(f^b)$$

where $\{e^i\}$ and $\{f^a\}$ are the dual coframes for the vertical and horizontal codistributions $\mathcal{V}^*$ and $\mathcal{H}^*$ of the Riemannian submersion; we adopt the Einstein convention and sum over repeated indices. Let d denote exterior differentiation and δ the adjoint operator, codifferentiation. We refer to [10–12, 22] for the proof of the following result.

Theorem 2.1. Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds.
(a) We have $\delta_Z \pi^* - \pi^* \delta_Y = \{\mathcal{E} + \mathrm{int}(\theta)\}\pi^*$.
(b) Let $p = 0$. Then the following conditions are equivalent.
 (i) We have $\Delta^0_Z \pi^* = \pi^* \Delta^0_Y$.
 (ii) For all $\lambda \in \mathbb{R}$, $\exists \mu(\lambda) \in \mathbb{R}$ so $\pi^* E(\lambda, \Delta^0_Y) \subseteq E(\mu(\lambda), \Delta^0_Z)$.
 (iii) The fibres of π are minimal.
(c) Let $1 \leqslant p \leqslant \dim(Y)$. Then the following conditions are equivalent.
 (i) We have $\Delta^p_Z \pi^* = \pi^* \Delta^p_Y$.
 (ii) For all $\lambda \in \mathbb{R}$, $\exists \mu(\lambda) \in \mathbb{R}$ so $\pi^* E(\lambda, \Delta^p_Y) \subseteq E(\mu(\lambda), \Delta^p_Z)$.
 (iii) The fibres of π are minimal and the horizontal distribution of π is integrable.
(d) Assume there exists $\Phi_p \in E(\lambda, \Delta^p_Y)$ so that $\pi^* \Phi_p \in E(\mu, \Delta^p_Z)$ for $\lambda \neq \mu$. Then $p \neq 0$ and $\lambda < \mu$.

Definition 2.2. The Hopf fibration. Let $\mathbb{H}$ be the quaternions. Let S^3 be the unit sphere in $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}$; S^3 is a group under quaternion multiplication. If $\boldsymbol{x} \in S^3$, let

$$\boldsymbol{x} = (x^0, x^1, x^2, x^3) = (z^0, z^1) = x^0 + x^1 i + x^2 j + x^3 k$$

where $z^0 = x^0 + \mathrm{i}x^1$ and $z^1 = x^2 + \mathrm{i}x^3$. Let $\partial_a = \partial/\partial x^a$. Let $\zeta_1(\boldsymbol{x}) = i \cdot \boldsymbol{x}$, $\zeta_2(\boldsymbol{x}) = j \cdot \boldsymbol{x}$, and $\zeta_3(\boldsymbol{x}) = k \cdot \boldsymbol{x}$. The vectors $\{\boldsymbol{x}, \zeta_1(\boldsymbol{x}), \zeta_2(\boldsymbol{x}), \zeta_3(\boldsymbol{x})\}$ form an orthonormal basis for $\mathbb{R}^4$ so $\{\zeta_1, \zeta_2, \zeta_3\}$ is an orthonormal frame for TS^3. Let $\{\zeta^1, \zeta^2, \zeta^3\}$ be the dual coframe for the cotangent bundle T^*S^3. The ζ^i and ζ_i are invariant under *right* multiplication. We compute

$$\begin{aligned}
&\zeta_1 = -x^1\partial_0 + x^0\partial_1 - x^3\partial_2 + x^2\partial_3, \ [\zeta_2, \zeta_3] = -2\zeta_1\\
&\zeta_2 = -x^2\partial_0 + x^3\partial_1 + x^0\partial_2 - x^1\partial_3, \ [\zeta_3, \zeta_1] = -2\zeta_2\\
&\zeta_3 = -x^3\partial_0 - x^2\partial_1 + x^1\partial_2 + x^0\partial_3, \ [\zeta_1, \zeta_2] = -2\zeta_3\\
&\zeta^1 = -x^1\,\mathrm{d}x^0 + x^0\,\mathrm{d}x^1 - x^3\,\mathrm{d}x^2 + x^2\,\mathrm{d}x^3, \ \mathrm{d}\zeta^1 = 2\zeta^2 \wedge \zeta^3\\
&\zeta^2 = -x^2\,\mathrm{d}x^0 + x^3\,\mathrm{d}x^1 + x^0\,\mathrm{d}x^2 - x^1\,\mathrm{d}x^3, \ \mathrm{d}\zeta^2 = 2\zeta^3 \wedge \zeta^1\\
&\zeta^3 = -x^3\,\mathrm{d}x^0 - x^2\,\mathrm{d}x^1 + x^1\,\mathrm{d}x^2 + x^0\,\mathrm{d}x^3, \ \mathrm{d}\zeta^3 = 2\zeta^1 \wedge \zeta^2.
\end{aligned}$$

We let the circle $S^1 \subset \mathbb{C}$ act on S^3 by complex multiplication from the left; let π be the natural projection from S^3 to the quotient manifold $S^1\backslash S^3 = \mathbb{C}P^1 = S^2$; this is the Hopf fibration. In terms of coordinates, $\pi : S^3 \to S^2$ is defined by

$$\begin{aligned}\pi(\boldsymbol{x}) &= (2\,\mathrm{Re}(z^0\bar{z}^1), 2\,\mathrm{Im}(z^0\bar{z}^1), |z^0|^2 - |z^1|^2) \\ &= (2(x^0x^2 + x^1x^3), 2(x^1x^2 - x^0x^3), x^0x^0 + x^1x^1 - x^2x^2 - x^3x^3).\end{aligned}$$

Since $\boldsymbol{x} \to \mathrm{e}^{\mathrm{i}t} \cdot \boldsymbol{x}$ defines the one-parameter flow for the vector field ζ_1, $\pi_*\zeta_1 = 0$. If $y = (y^0, y^1, y^2)$ are the standard coordinates on $\mathbb{R}^3$, then

$$\pi^*(\mathrm{d}y^0 \circ \mathrm{d}y^0 + \mathrm{d}y^1 \circ \mathrm{d}y^1 + \mathrm{d}y^2 \circ \mathrm{d}y^2) = 4\zeta^2 \circ \zeta^2 + 4\zeta^3 \circ \zeta^3.$$

We let g_n be the standard metric on S^n. Then π_* is a Riemannian submersion from (S^3, g_3) to $(S^2, \frac{1}{4}g_2)$ with vertical distribution spanned by ζ_1 and horizontal distribution spanned by $\{\zeta_2, \zeta_3\}$. We can check this normalization by computing $2\pi^2 = \mathrm{vol}(S^3, g_3) = \pi \cdot (2\pi) = \mathrm{vol}(S^2, \frac{1}{4}g_2) \cdot \mathrm{vol}(S^1, g_1)$. We have that $\pi^*\nu_2 = \zeta^2 \wedge \zeta^3$. Note that although the vector fields $\{\zeta_2, \zeta_3\}$ and covector fields $\{\zeta^2, \zeta^3\}$ are horizontal, they are *not* horizontal lifts of vector and covector fields on S^2.

Theorem 2.3. (a) Let $\pi : (S^3, \tilde{g}_3) \to (S^2, \frac{1}{4}g_2)$ be a Riemannian submersion where $\tilde{g}_3$ is a right invariant metric on S^3. There exist constants $\alpha > 0$ and $\boldsymbol{\beta} = (\beta_2, \beta_3)$ so that $\tilde{g}_3 = \tilde{g}_3(\alpha, \beta_2, \beta_3) = \chi^1 \circ \chi^1 + \chi^2 \circ \chi^2 + \chi^3 \circ \chi^3$ for

$$\begin{aligned}&\chi_1 := \alpha^{-1}\zeta_1 \qquad \chi_2 := \zeta_2 + \beta_2\zeta_1 \qquad \chi_3 := \zeta_3 + \beta_3\zeta_1 \\ &\chi^1 := \alpha(\zeta^1 - \beta_2\zeta^2 - \beta_3\zeta^3) \qquad \chi^2 := \zeta^2 \qquad \chi^3 := \zeta^3.\end{aligned}$$

(b) Assume $\boldsymbol{\beta} = 0$. Then the fibres of π are geodesics and
 (i) We have $\pi^*\Phi_0 \in E(\lambda, \Delta^0_{S^3}) \Leftrightarrow \Phi_0 \in E(\lambda, \Delta^0_{S^2})$.
 (ii) We have $\pi^*\Phi_1 \in E(\lambda, \Delta^1_{S^3}) \Leftrightarrow \Phi_1 = \mathrm{d}\Phi_0$ for $\Phi_0 \in E(\lambda, \Delta^0_{S^2})$.
 (iii) We have $\pi^*\Phi_2 \in E(\lambda, \Delta^2_{S^3}) \Leftrightarrow \Phi_2 = c\nu_2$ and $\lambda = 4\alpha^2$.

(c) Let $\boldsymbol{\beta} \neq 0$. Then the fibres of π are not geodesics. If $\pi^*\Phi_p \in E(\lambda, \Delta^p_{S^3})$, then $p = 0$, $\Phi_0 = c$, and $\lambda = 0$.

Remark 2.4. The metrics $\tilde{g}_3(\alpha, 0, 0)$ are a smooth one-parameter family of Riemannian submersions where the eigenvalue changes from 0 to any real value. These metrics are generalizations of the metrics considered in [21], example 4.3.

Proof. We compute

$$[\chi_1, \chi_2] = -2\alpha^{-1}\chi_3 + 2\beta_3\chi_1 \qquad [\chi_3, \chi_1] = -2\alpha^{-1}\chi_2 + 2\beta_2\chi_1$$

$$[\chi_2, \chi_3] = -2\alpha(1 + |\boldsymbol{\beta}|^2)\chi_1 + 2\beta_2\chi_2 + 2\beta_3\chi_3$$

$$\mathrm{d}\chi^1 = 2\alpha(1 + |\boldsymbol{\beta}|^2)\chi^2 \wedge \chi^3 - 2\beta_2\chi^3 \wedge \chi^1 - 2\beta_3\chi^1 \wedge \chi^2$$

$$\mathrm{d}\chi^2 = 2\alpha^{-1}\chi^3 \wedge \chi^1 - 2\beta_2\chi^2 \wedge \chi^3$$

$$\mathrm{d}\chi^3 = 2\alpha^{-1}\chi^1 \wedge \chi^2 - 2\beta_3\chi^2 \wedge \chi^3$$

$$\mathrm{d}(\chi^1 \wedge \chi^2) = 0 \qquad \mathrm{d}(\chi^2 \wedge \chi^3) = 0 \qquad \mathrm{d}(\chi^3 \wedge \chi^1) = 0$$

$$\begin{aligned}\theta &= \tilde{g}_3(\nabla_{\chi_1}\chi_1, \chi_2)\chi_2 + \tilde{g}_3(\nabla_{\chi_1}\chi_1, \chi_3)\chi_3 \\ &= -\tilde{g}_3(\chi_1, [\chi_1, \chi_2])\chi_2 - \tilde{g}_3(\chi_1, [\chi_1, \chi_3])\chi_3 = -2\beta_3\chi_2 + 2\beta_2\chi_3\end{aligned}$$

$$\omega(\chi_1, \chi_2, \chi_3) = \tfrac{1}{2}\tilde{g}_3(\chi_1, [\chi_2, \chi_3]) = -\alpha(1 + |\boldsymbol{\beta}|^2)$$

$$\mathcal{E} = -2\alpha(1 + |\boldsymbol{\beta}|^2)\,\mathrm{ext}(\chi^1)\,\mathrm{int}(\chi^2)\,\mathrm{int}(\chi^3).$$

Since the fibres of π are one dimensional, they are minimal if and only if they are geodesics; this happens if and only if $\boldsymbol{\beta} = 0$.

Let $0 \neq \phi_p = \pi^*\Phi_p \in E(\mu, \Delta^p_{S^3})$ for some p. Suppose first that $p = 0$ and that Φ_0 is non-constant. We use theorem 2.1 to compute

$$\Delta^0_{S^3}\phi_0 - \pi^*\Delta^0_{S^2}\Phi_0 = -2\beta_3\chi_2\phi_0 + 2\beta_2\chi_3\phi_0 = (-2\beta_3\zeta_2 + 2\beta_2\zeta_3)\phi_0.$$

Suppose that $\boldsymbol{\beta} = (\beta_2, \beta_3) \neq 0$. We note that the push forward of $\beta_2\zeta_3 - \beta_3\zeta_2$ ranges over the circle of radius $4|\boldsymbol{\beta}|^2$ in T_yS^2 as $z \in \pi^{-1}(y)$. Since $\Delta^0_{S^3}\phi_0 = \mu\phi_0$, $2(\beta_2\zeta_3 - \beta_3\zeta_2)\phi_0$ is the pull-back of a function from S^2. Thus $(\beta_2\zeta_3 - \beta_3\zeta_2)\phi_0$ must vanish identically and hence all the derivatives of Φ_0 are constant contrary to our assumption. If $\boldsymbol{\beta} = 0$, then the fibres of π are totally geodesic and π^* intertwines $\Delta^0_{S^2}$ and $\Delta^0_{S^3}$ by theorem 2.1; this handles the case $p = 0$.

Suppose next that $p = 2$. Let $\Phi_2 = \Phi_0\nu_2$, let $\phi_2 = \pi^*\Phi_2$, and let $\phi_0 = \pi^*\Phi_0$. We use theorem 2.1 to see $\delta_{S^3}\phi_2 - \pi^*\delta_{S^2}\Phi_2 = \phi_0(-2\beta_2\chi^2 - 2\beta_3\chi^3 + 2\alpha(1 + |\boldsymbol{\beta}|^2)\chi^1)$ and

$$\begin{aligned}\Delta^2_{S^3}\phi_2 - \pi^*\Delta^2_{S^2}\Phi_2 = {} & (-2\beta_3\zeta_2\phi_0 + 2\beta_2\zeta_3\phi_0 + \mathcal{E}_{23}\phi_0)\chi^2 \wedge \chi^3 \\ & +(2\alpha(1 + |\boldsymbol{\beta}|^2)\zeta_3\phi_0 + \mathcal{E}_{31}\phi_0)\chi^3 \wedge \chi^1 - (2\alpha(1 + |\boldsymbol{\beta}|^2)\zeta_2\phi_0 + \mathcal{E}_{12}\phi_0)\chi^1 \wedge \chi^2\end{aligned}$$

for a suitably chosen constant $\mathcal{E}_{ij}$. Because ϕ_2 is an eigenform of $\Delta^2_{S^3}$, the left-hand side of this equation is the pull-back of a 2-form from the base and consequently the coefficient of $\chi^1 \wedge \chi^2$ vanishes. This shows $2\alpha(1 + |\boldsymbol{\beta}|^2)\zeta_2\phi_0 = -\mathcal{E}_{12}\phi_0$. The same argument as that given for functions shows that ϕ_0 is constant so we may assume $\phi_2 = \nu_2$ so $\delta_{S^2}\nu_2 = 0$ and

$$\begin{aligned}\delta_{S^3}\pi^*\nu_2 &= -2\beta_2\chi^2 - 2\beta_3\chi^3 + 2\alpha(1 + |\boldsymbol{\beta}|^2)\chi^1 \\ &= 2\alpha^2(1 + |\boldsymbol{\beta}|^2)\zeta^1 - 2(1 + \alpha^2(1 + |\boldsymbol{\beta}|^2))(\beta_2\zeta^2 + \beta_3\zeta^3).\end{aligned}$$

The derivative of this must involve only horizontal terms; thus the coefficients of ζ^2 and ζ^3 must vanish. This implies $\boldsymbol{\beta} = 0$ and $\Delta^2_{S^3}\pi^*\nu_2 = 4\alpha^2\pi^*\nu_2$. This handles the case $p = 2$.

Finally, we discuss the case $p = 1$. Let $\phi_1 = \pi^*\Phi_1 \in E(\mu, \Delta^1_{S^3})$; $\mu \neq 0$ as the first de Rham cohomology group $H^1(S^3; \mathbb{C})$ vanishes. Let $\Phi_2 = \mathrm{d}\Phi_1$ and $\phi_2 = \mathrm{d}\phi_1 = \pi^*\Phi_2$. Since we have that

$$\mathrm{d}\delta\phi_2 = \mathrm{d}\delta\,\mathrm{d}\phi_1 = \mathrm{d}(\delta\,\mathrm{d} + \mathrm{d}\delta)\phi_1 = \mu\,\mathrm{d}\phi_1 = \mu\phi_2$$

we can apply (c) to see that $\mathrm{d}\phi_1 = c\pi^*\nu_2$. This means $\mathrm{d}\Phi_1 = c\nu_2$ and hence $c = 0$. Thus $\mathrm{d}\phi_1 = 0$. Let $\phi_0 = \mu^{-1}\delta\phi_1 \in C^\infty S^3$. Then

$$\mathrm{d}\phi_0 = \mu^{-1}\,\mathrm{d}\delta\phi_1 = \mu^{-1}\Delta^1_{S^3}\phi_1 = \phi_1.$$

Consequently, $0 = \phi_1(\zeta_1) = \mathrm{d}\phi_0(\zeta_1) = \zeta_1(\phi_0)$ so ϕ_0 is constant on the fibre circles. This implies $\phi_0 = \pi^*\Phi_0$ for $\Phi_0 \in C^\infty S^2$. Furthermore

$$\Delta^0_{S^3}\phi_0 = \delta\,\mathrm{d}\phi_0 = \mu^{-1}\delta\mathrm{d}\,\delta\phi_1 = \mu^{-1}\delta\Delta^1_{S^3}\phi_1 = \delta\phi_1 = \mu\phi_0.$$

If ϕ_0 is constant, $0 = \mathrm{d}\phi_0 = \phi_1$ which is false. If ϕ_0 is not constant, then $\boldsymbol{\beta} = 0$. □

3. Principal bundles

In this section, we discuss generalizations of theorem 2.3 to higher dimensions. We begin by recalling some basic definitions and refer to Eguchi *et al* [9] for further details. Let G be a compact Lie group and let g_G be a bi-invariant Riemannian metric on G. Let $G \to P \xrightarrow{\pi} Y$ be a principal bundle over a closed manifold Y. We choose a metric g_P on P so that g_P is invariant under the G action, so that the restriction of g_P to the fibres is g_G, and so that $\pi : (P, g_P) \to (Y, g_Y)$ is a Riemannian submersion. The horizontal distribution

$\mathcal{H}$ of π is invariant under the G action and defines a principal connection on P. Such a metric will be called a *bundle* metric and the fibres are necessarily totally geodesic. For example, the metric $\tilde{g}_3(\alpha, \beta_2, \beta_3)$ on the Hopf bundle $S^1 \to S^3 \to S^2$ discussed in section 2 is a bundle metric if and only if $\beta_2 = \beta_3 = 0$.

Let g_P be a bundle metric. Let $y = (y^a)$ be a system of local coordinates on Y and let s be a local section to P. The map $(g, y) \to s(y) \cdot g$ gives local coordinates on P. Let $\{\mathcal{A}_i\}$ be a basis for the right-invariant vector fields on G and let $\{\mathcal{A}^i\}$ be the corresponding dual basis for the right-invariant covector fields on G. The horizontal distribution $\mathcal{H}$ is spanned by vector fields of the form $\chi_a := \partial_a^y + \Gamma_a^i(y)\mathcal{A}_i$ and the vertical distribution is spanned by the $\mathcal{A}_i$. The connection 1-form of the principal connection is a Lie-algebra valued 1-form given by $A := \Gamma_a^i(y)\,\mathrm{d}y^a \otimes \mathcal{A}_i$. The curvature $\mathcal{F}$ of the principal bundle is the Lie-algebra valued 2-form given by

$$\mathcal{F} := \mathrm{d}y^a \wedge \mathrm{d}y^b \otimes (\partial_a \Gamma_b^i \mathcal{A}_i - \partial_b \Gamma_a^i \mathcal{A}_i + \Gamma_a^i \Gamma_b^j [\mathcal{A}_i, \mathcal{A}_j]) = g^{ij}\omega_{abi}\,\mathrm{d}y^a \wedge \mathrm{d}y^b \otimes \mathcal{A}_j.$$

Thus the tensor ω defined in section 2 is the curvature of the principal connection on P.

Let $SO(n)$ be the special orthogonal group. Let $e_1 := (1, 0, \ldots, 0) \in S^{n-1}$. If $\sigma \in SO(n)$, let $\pi : \sigma \to e_1 \cdot \sigma$ define the principal bundle $SO(n-1) \to SO(n) \xrightarrow{\pi} S^{n-1}$. The bi-invariant metric g_{SO} on $SO(n)$ is unique up to scale; we choose the normalizing constant so that $\pi : (SO(n), g_{SO}) \to (S^{n-1}, g_{n-1})$ is a Riemannian submersion with respect to the standard metric g_{n-1} on the sphere; g_{SO} is a bundle metric.

Theorem 3.1. Let $\tilde{g}_{SO}$ and $\tilde{g}_{n-1}$ be metrics on $SO(n)$ and on S^{n-1} so that π is a Riemannian submersion from $(SO(n), \tilde{g}_{SO})$ to $(S^{n-1}, \tilde{g}_{n-1})$. Assume $n \geqslant 4$.

(a) We do not assume $\tilde{g}_{SO}$ is a bundle metric. Let $\tilde{\nu}_{n-1}$ be the volume element of S^{n-1} defined by the metric $\tilde{g}_{n-1}$. If $\pi^*\tilde{\nu}_{n-1} \in E(\mu, \Delta^{n-1}_{SO(n)})$, then $\mu = 0$ and $n = 4$ or $n = 8$.

(b) Let g_{n-1} and g_{SO} be the standard metrics.

(i) Suppose that $\Phi_0 \in E(\lambda, \Delta^0_{S^{n-1}})$. Then we have that $\pi^*\Phi_0 \in E(\lambda, \Delta^0_{SO(n)})$, $\mathrm{d}\Phi_0 \in E(\lambda, \Delta^1_{S^{n-1}})$, and $\pi^*\,\mathrm{d}\Phi_0 \in E(\lambda, \Delta^1_{SO(n)})$.

(ii) Suppose that $\Phi_p \in E(\lambda, \Delta^p_{S^{n-1}})$ and that $\pi^*\Phi_p \in E(\mu, \Delta^p_{SO(n)})$. Then $\lambda = \mu$ and either $p = 0$ or $p = 1$. If $p = 1$, then $\Phi_1 = \mathrm{d}\Phi_0$ for some $\Phi_0 \in E(\mu, \Delta^0_{S^{n-1}})$.

Remark 3.2. If $n = 3$, then $SO(3) = \mathbb{R}P^3$ and we have $S^1 \to \mathbb{R}P^3 \to S^2$; this Riemannian submersion was studied by the third author in [21]. Let $SPIN(n)$ be the spinor group; $SPIN(n)$ is the universal cover of $SO(n)$ for $n \geqslant 3$. The double cover $\mathbb{Z}_2 \to SPIN(n) \to SO(n)$ then lets us construct a Riemannian submersion $SPIN(n-1) \to SPIN(n) \to S^{n-1}$. If $n = 3$, $SPIN(3) = S^3$ and we recover the Hopf fibration $S^1 \to S^3 \xrightarrow{\pi} S^2$ discussed in section 2. Theorem 3.1 extends immediately to the spinor groups.

We will derive theorem 3.1 from the following more general result which shows that eigenvalues cannot change if $H^1(G; \mathbb{C}) = 0$.

Theorem 3.3. Let G be a compact Lie group with $H^1(G; \mathbb{C}) = 0$. We assume that $\pi : (P, g_P) \to (Y, g_Y)$ is a Riemannian submersion where $G \to P \xrightarrow{\pi} Y$ is a principal bundle.

(a) We do not assume g_P is a bundle metric. If $0 \neq \Phi_p \in E(0, \Delta^p_Y)$ and if $\pi^*\Phi_p \in E(\mu, \Delta^p_P)$, then $\mu = 0$. If in addition, Φ_p is the volume form on Y, then the mean curvature vector θ and the curvature ω of π vanish.

(b) We do assume that g_P is a bundle metric. If $0 \neq \Phi_p \in E(\lambda, \Delta^p_Y)$ and if $\pi^*\Phi_p \in E(\mu, \Delta^p_P)$, then $\lambda = \mu$. If $\mathrm{d}\Phi_p = 0$, then $\mathcal{E}\pi^*\Phi_p = 0$.

Proof. Let $\{F^a\}$ be a local orthonormal coframe on Y. We take a local product decomposition of P. If $A = \{1 \leqslant a_1 < \ldots < a_q \leqslant \dim(Y)\}$ is a multi-index, we let $F^A = F^{a_1} \wedge \ldots \wedge F^{a_q}$; the $\{F^A\}$ for $|A| = q$ are a local orthonormal frame for $\Lambda^q Y$.

Suppose the hypothesis of (a) holds. Since Φ_p is harmonic, we may apply theorem 2.1 to see $\mu\pi^*\Phi_p = \mathrm{d}\psi_{p-1}$ for

$$\psi_{p-1} := (\mathrm{int}(\theta) + \omega_{abi}\,\mathrm{ext}(e^i)\,\mathrm{int}(f^a)\,\mathrm{int}(f^b))\pi^*\Phi_p.$$

We average over the action of the fibre G with respect to Haar measure to define

$$\tilde\psi_{p-1} := \int_{g\in G}(g^*\psi_{p-1})\,\mathrm{d}g \qquad \mu\pi^*\Phi_p = \mathrm{d}\tilde\psi_{p-1}.$$

We may express $\psi_{p-1} = \Sigma_{|A|=p-2}\alpha_A\pi^*F^A$ where the α_A are suitably chosen 1-forms on P. Let

$$\beta_A := \int_{g\in G} g^*\alpha_A\,\mathrm{d}g \qquad \tilde\psi_{p-1} = \Sigma_{|A|=p-2}\beta_A\pi^*F^A.$$

Since β_A is G invariant, we may decompose $\beta_A = \pi^*\gamma_A + \Xi_A$ where $\gamma_A \in C^\infty\Lambda^1 Y$ and where $\Xi_A = \Xi_A(y)$ is an invariant 1-form on the fibre with coefficients which depend on the base. Since $\mathrm{d}\tilde\psi_{p-1} = \mu\pi^*\Phi_p$ is the pull-back of a form from the base, the terms in $\mathrm{d}\tilde\psi_{p-1}$ which involve vertical 2-forms must vanish. This implies the restriction of $\mathrm{d}\Xi_A$ to the fibre vanishes. Since $H^1(G;\mathbb{C}) = 0$, there are no non-trivial closed invariant 1-forms on the fibre. This shows that $\Xi_A = 0$ and therefore $\tilde\psi_{p-1}$ does not involve the fibre coordinate. Thus there exists $\Psi_{p-1} \in C^\infty\Lambda Y$ so $\tilde\psi_{p-1} = \pi^*\Psi_{p-1}$. This implies that $\mu\Phi_p = \mathrm{d}\Psi_{p-1}$ and hence $\mu = 0$. The equations $\psi_{p-1} = \delta\pi^*\Phi_p$ and $\mathrm{d}\psi_{p-1} = \mu\pi^*\Phi_p = 0$ imply

$$0 = (\mathrm{d}\psi_{p-1}, \pi^*\Phi_p)_{L^2(P)} = (\psi_{p-1}, \delta\pi^*\Phi_p)_{L^2(P)} = (\psi_{p-1}, \psi_{p-1})_{L^2(P)}$$

so $\psi_{p-1} = 0$. We decompose ψ_{p-1} into horizontal and vertical components to see $\mathrm{int}(\theta)\pi^*\Phi_p = 0$ and $\mathcal{E}\pi^*\Phi_p = 0$. If Φ_p is the volume form on Y, it then follows that $\theta = 0$ and $\omega = 0$. This completes the proof of the first assertion.

Next suppose that the hypothesis of (b) holds. Since g_P is a bundle metric, the fibres of π are totally geodesic so $\theta = 0$. Suppose first $\mathrm{d}\Phi_p = 0$ so $\mathrm{d}\pi^*\Phi_p = 0$. As in the proof of (a), we expand

$$\mathcal{E}\pi^*\Phi_p = (\delta_P\pi^* - \pi^*\delta_Y)\Phi_p = \beta_A \wedge \pi^*F^A$$

where the β_A are vertical covectors. Since g_P is a bundle metric, the construction is G equivariant so the β_A are G invariant. Since $\theta = 0$ and $\mathrm{d}\Phi_p = 0$, we have that $\mathrm{d}\mathcal{E}\pi^*\Phi_p = \Delta_P^p\pi^*\Phi_p - \pi^*\Delta_Y^p\Phi_p = (\mu-\lambda)\pi^*\Phi_p$ has no vertical dependence. Thus the vertical derivative of the restriction of β_A to the fibres vanishes. Since $H^1(G;\mathbb{C}) = 0$ and since β_A is G invariant, this implies the restriction of β_A to the fibres vanishes and hence $\beta_A = 0$ as $\beta_A \in \mathcal{V}^*$. Consequently $\mathcal{E}\pi^*\Phi_P = 0$ and $\mu = \lambda$. Now suppose that $\mathrm{d}\Phi_p \neq 0$. Since $\pi^*\,\mathrm{d}\Phi_p = \mathrm{d}\pi^*\Phi_p \in E(\mu, \Delta_Y^{p+1})$, we may replace Φ_p by $\mathrm{d}\Phi_p$ to conclude $\lambda = \mu$ and complete the proof. $\square$

Proof. We can now prove theorem 3.1(a). Let $\pi : (SO(n), \tilde g_{SO}) \to (S^{n-1}, \tilde g_{n-1})$ be a Riemannian submersion where we do not assume $\tilde g_{SO}$ is a bundle metric. Let $\tilde\nu_{n-1}$ be the volume form on S^{n-1} with respect to the metric $\tilde g_{n-1}$; $\tilde\nu_{n-1}$ is harmonic. We suppose $\pi^*\tilde\nu_{n-1} \in E(\mu, \Delta^{n-1}_{SO(n)})$. Since $n \geqslant 4$, $\pi_1 SO(n-1) = \mathbb{Z}_2$ and $H^1(SO(n-1);\mathbb{C}) = 0$. Thus we may apply theorem 3.3 to see that $\mu = 0$, $\theta = 0$, and $\omega = 0$. Since S^{n-1} is simply connected, the vanishing of the curvature implies that the fibration has a global section, see for example [10]. This implies that S^{n-1} is parallelizable and therefore $n-1 = 1, 3, 7$; see for example [2, 6]. $\square$

Remark 3.4. If $n = 4$, let $\mathbb{F}$ be the quaternions and if $n = 8$, let $\mathbb{F}$ be the Cayley numbers. Let $\{e_i\}$ be a basis for $\mathbb{F}$ where $e_1 = 1$. Then the map $x \to (e_1x, \ldots, e_nx)$ is a section to the map π and defines a splitting $SO(n) = SO(n-1) \times S^{n-1}$ such that π is projection on the second factor. We use this splitting to define a product metric on $SO(n)$; $\pi^*\nu_{n-1}$ is harmonic for such a metric. Thus theorem 3.1 is sharp. This gives a non-standard horizontal structure for π with zero curvature; $\pi^*\nu_{n-1}$ is not harmonic with respect to the standard bi-invariant metric on $SO(n)$. We can describe such a structure if $n = 4$ which is right invariant as follows. Let $\{e_i\}$ be the standard orthonormal basis for $\mathbb{R}^n$. Let $so(n)$ be the Lie algebra of $SO(n)$; we identify $so(n)$ with the set of skew symmetric $n \times n$ matrices. We take elements A_{ij} of $so(n)$ for $1 \leqslant i, j \leqslant n$ where A_{ij} is the skew-adjoint transformation $A_{ij} : e_j \to e_i \to -e_j$. The Lie algebra structure is given by $[A_{ij}, A_{k\ell}] = \delta_{jk}A_{i\ell} - \delta_{j\ell}A_{ik} - \delta_{ik}A_{j\ell} + \delta_{i\ell}A_{jk}$. Let $\mathcal{H} = \mathrm{span}\{\chi_{14}, \chi_{24}, \chi_{34}\}$ define the horizontal subspace where

$$\chi_{14} := A_{14} + A_{23} \qquad \chi_{24} := A_{24} - A_{13} \qquad \chi_{34} := A_{34} + A_{12}.$$

The vertical space $\mathcal{V} = \mathrm{span}\{A_{13}, A_{23}, A_{12}\} = so(3)$. Then $g(A_{ab}, [A_{ab}, \chi_{cn}]) = 0$ for $1 \leqslant a, b, c \leqslant 3$ so $\theta = 0$ and the fibres are minimal. Furthermore

$$[\chi_{14}, \chi_{24}] = -2\chi_{34} \qquad [\chi_{24}, \chi_{34}] = -2\chi_{14} \qquad [\chi_{34}, \chi_{14}] = -2\chi_{24}$$

so the horizontal distribution is integrable. Thus by theorem 2.1, π^* intertwines the eigenspaces of the Laplacian.

Proof. We can now prove theorem 3.1(b). Let g_{SO} be the bi-invariant metric on $SO(n)$ and let g_{n-1} be the standard metric on S^{n-1}. Assertion b(i) is immediate from theorem 2.1 since the fibres of π are totally geodesic in this situation. Let $\Phi_p \in E(\lambda, \Delta^p_{S^{n-1}})$ and $\pi^*\Phi_p \in E(\mu, \Delta^p_{SO(n)})$. Since $n \geqslant 4$ and since the metric is $SO(n-1)$ invariant, we may apply theorem 3.3 (b) to see $\lambda = \mu$.

Suppose first that $\mathrm{d}\Phi_p = 0$ and that $p \geqslant 2$. We compute that

$$\mathcal{E} = \Sigma_{1 \leqslant a<b<n}\, \mathrm{ext}(A^{ab})\, \mathrm{int}(A^{an})\, \mathrm{int}(A^{bn}).$$

Fix a point $\boldsymbol{x}_0 \in S^{n-1}$ where $\Phi_p(\boldsymbol{x}_0) \neq 0$. Since the $\{A^{an}\}$ are basis for $\mathcal{H}^*$ and since $p \geqslant 2$, there exist a, b so $\mathrm{ext}(A^{ab})\, \mathrm{int}(A^{an})\, \mathrm{int}(A^{bn})\Phi_p(\boldsymbol{x}) \neq 0$. By theorem 3.3(b), $\mathcal{E}\pi^*\Phi_p = 0$ so this is impossible.

Suppose next that $\mathrm{d}\Phi_p \neq 0$ and that $p \geqslant 1$. Replacing Φ_p by $\mathrm{d}\Phi_p$ and p by $p+1$, we again arrive at a contradiction using the argument of the previous paragraph. We conclude therefore either that $p = 0$ or that $p = 1$ and $\mathrm{d}\Phi_p = 0$. Suppose that $p = 1$ and $\mathrm{d}\Phi_1 = 0$. Since $H^1(S^{n-1}; \mathbb{C}) = 0$, $\mu > 0$. Let $\Phi_0 = \mu^{-1}\delta_Y\Phi_1$. Then $\Phi_0 \in E(\mu, \Delta^0_{S^{n-1}})$ and $\mathrm{d}\Phi_0 = \Phi_1$. □

Theorem 3.3 shows that eigenvalues do not change for a principal bundles-with-structure group G if $H^1(G; \mathbb{C}) = 0$ and if the metric on P is a bundle metric. In fact theorem 3.3 is sharp. Let ζ^a be the basis for the Lie algebra of S^3 discussed above; recall that $\zeta^2 \wedge \zeta^3 \in E(4, \Delta^2_{S^3})$.

Theorem 3.5. Let G be a compact connected Lie group with $H^1(G; \mathbb{C}) \neq 0$. Let $\pi : P := G \times S^3 \to S^3$ be projection on the second factor. Let g_G be a bi-invariant metric on G. For any $\epsilon \in \mathbb{R}$, there exists a bundle metric g_ϵ on P so that $\pi^*(\zeta^2 \wedge \zeta^3) \in E(4\epsilon^2 + 4, \Delta^2_P)$.

Proof. Choose an orthonormal basis Ξ^i for the Lie algebra of G so that $d\Xi^1 = 0$. Then Ξ^1 is harmonic. Since G is connected, it acts trivially on the harmonic spaces and hence Ξ^1 is bi-invariant. Define

$$ds_\epsilon^2 = (\Xi^1 + \epsilon\zeta^1) \circ (\Xi^1 + \epsilon\zeta^1) + \Sigma_{i\geqslant 2}\Xi^i \circ \Xi^i + \Sigma_a \zeta^a \circ \zeta^a.$$

The restriction of ds_ϵ^2 to the fibres of π is g_G; since Ξ^1 is bi-invariant, g_ϵ is invariant under the action of G on P and is a bundle metric. Let Ξ_i and ζ_a be the corresponding dual-vector fields. Then $\{\Xi_i\}$ span the vertical space $\mathcal{V}$ and $\{\zeta_1 - \epsilon\Xi_1, \zeta_2, \zeta_3\}$ span the horizontal space $\mathcal{H}$ of $\pi : (P, g_\epsilon) \to (S^3, g_3)$. Since $g_\epsilon(\Xi_1, [\zeta_2, \zeta_3]) = -2\epsilon$

$$\mathcal{E} = -2\epsilon\, \mathrm{ext}(\Xi^1 + \epsilon\zeta^1)\, \mathrm{int}(\zeta^2)\, \mathrm{int}(\zeta^3).$$

Let $\Phi_2 = \zeta^2 \wedge \zeta^3$. Since $d\Xi^1 = 0$ and $d\Phi_2 = 0$

$$(\Delta_P^2 \pi^* - \pi^* \Delta_{S^3}^2)\Phi_2 = 2\epsilon\, d(\Xi^1 + \epsilon\zeta^1) = 4\epsilon^2 \Phi_2. \qquad \square$$

Remark 3.6. We identify $\mathbb{R}^{2k} = \mathbb{C}^k$ to embed $S^{2k-1} \subset \mathbb{C}^k$ for $k \geqslant 2$. If $\boldsymbol{x} \in S^{2k-1}$, let $\zeta_1(\boldsymbol{x}) = i \cdot \boldsymbol{x}$ define a non-vanishing vector field on S^{2k-1}. Let ζ^1 be the corresponding co-vector field where we use the standard metric g_{2k-1} on S^{2k-1}. If $\sigma : S^{2k-1} \to \mathbb{C}P^{k-1}$ is the canonical projection from the sphere to complex projective space, and if ω_2 is the Kaehler form on $\mathbb{C}P^{k-1}$, then $\sigma^*\omega_2 = d\zeta^1$ modulo a suitable normalization and

$$\Phi_{2p} = (d\zeta^1)^p \in E(\lambda_{p,k}, \Delta_{S^{2k-1}}^{2p})$$

for $1 \leqslant p < k$ and for suitably chosen $\lambda_{p,k} > 0$. Let $P = G \times S^{2k-1}$ and let

$$ds_\epsilon^2 := (\Xi^1 + \epsilon\zeta^1) \circ (\Xi^1 \circ \epsilon\zeta^1) + \Sigma_{i\geqslant 2}\Xi^i \circ \Xi^i + ds^2_{S^{2k-1}}.$$

One can then verify that $\pi^*\Phi_{2p} \in E(\lambda_{p,k} + c_{p,k}\epsilon^2, \Delta_P^{2p})$ for suitably chosen positive constants $c_{p,k}$. This provides higher-dimensional examples and forms of higher degree where eigenvalues change. We omit details in the interests of brevity.

4. The complex Hopf fibration

We now turn to complex geometry. Let $w = (w^i)$ for $w^i := u^i + \sqrt{-1}v^i$ be local holomorphic coordinates on manifold M of complex dimension m. We define

$$J(\partial/\partial u^i) := \partial/\partial v^i \qquad J(\partial/\partial v^i) := -\partial/\partial u^i \qquad dw^i := du^i + \sqrt{-1}\,dv^i$$
$$d\bar{w}^i := du^i - \sqrt{-1}\,dv^i.$$

A Riemannian metric g_M is *Hermitian* if $g_M(X, Y) = g_M(JX, JY)$ for all real tangent vectors; we extend such a metric to the complexified tangent bundle to be complex linear in the first factor and conjugate linear in the second factor. Let

$$\Lambda^{p,q}M := \mathrm{span}_{|I|=p,|J|=q}\{dw^I \wedge d\bar{w}^J\}$$

where $dw^I := dw^{i_1} \wedge \ldots \wedge dw^{i_p}$ and $d\bar{w}^J := d\bar{w}^{j_1} \wedge \ldots \wedge d\bar{w}^{j_q}$. The almost complex structure J and the vector bundles $\Lambda^{p,q}M$ are invariantly defined. We decompose $d = \partial + \bar{\partial}$ and $\delta = \delta_1 + \delta_2$ for

$$\partial : C^\infty\Lambda^{p,q}M \to C^\infty\Lambda^{p+1,q}M \qquad \bar{\partial} : C^\infty\Lambda^{p,q}M \to C^\infty\Lambda^{p,q+1}$$
$$\delta_1 : C^\infty\Lambda^{p+1,q}M \to C^\infty\Lambda^{p,q}M \qquad \delta_2 : C^\infty\Lambda^{p,q+1}M \to C^\infty\Lambda^{p,q}$$

δ_1 is the adjoint of ∂ and δ_2 is the adjoint of $\bar{\partial}$. Let

$$\Delta_M^{p,q} = (\bar{\partial}\delta_2 + \delta_2\bar{\partial}) \qquad \text{on } C^\infty\Lambda^{p,q}M.$$

Let $\pi : Z \to Y$ be a Riemannian submersion. In the complex setting, we will always assume that Z and Y are complex manifolds, that π is holomorphic, and that the metrics on Z and on Y are Hermitian. The complexification of pull-back defines $\pi^* : C^\infty\Lambda^{p,q}Y \to C^\infty\Lambda^{p,q}Z$.

Let $Z := S^1 \times S^3$ be the Hopf manifold. Let ζ_0 and ζ^0 be the usual orthonormal basis for the tangent and cotangent bundles of the circle and let ζ_i and ζ^i be the right-invariant vector fields and covector fields defined in section 2. We give Z the product metric $\mathrm{d}s^2 = \zeta^0\circ\zeta^0 + \zeta^1\circ\zeta^1 + \zeta^2\circ\zeta^2 + \zeta^3\circ\zeta^3$. The Hopf manifold is a complex manifold which is not Kaehler. Let $J(\zeta_0) = \zeta_1$, $J(\zeta_1) = -\zeta_0$, $J(\zeta_2) = \zeta_3$, and $J(\zeta_3) = -\zeta_2$ define an almost complex structure on Z; J is unitary so $\mathrm{d}s^2$ is a Hermitian metric. The holomorphic tangent bundle is spanned by

$$\xi_0 := \tfrac{1}{2}(\zeta_0 - \sqrt{-1}\zeta_1) \qquad \text{and} \qquad \xi_1 := \tfrac{1}{2}(\zeta_2 - \sqrt{-1}\zeta_3)$$
$$4[\xi_0, \xi_1] = -\sqrt{-1}([\zeta_1, \zeta_2] - \sqrt{-1}[\zeta_1, \zeta_3]) = 2\sqrt{-1}\zeta_3 - 2\zeta_2 = -4\xi_1$$

so the Nirenberg–Neulander integrability condition is satisfied and (Z, J) is a complex manifold. This can also be seen directly; $Z = \mathbb{C}^2 - \{0\}/\mathbb{Z}$. We give S^2 the standard complex structure. Let $\tilde\pi(\lambda, \boldsymbol{x}) = \pi(\boldsymbol{x}) : Z \to S^2$ define the standard fibration from the Hopf manifold to S^2. Then $\tilde\pi_*\xi_0 = 0$ and $\tilde\pi_*\xi_1$ is a holomorphic tangent vector on S^2. Thus $\tilde\pi_*$ is a holomorphic Riemannian submersion and the metrics involved are Hermitian. Note that the metric on S^2 is Kaehler but that the metric on Z is not Kaehler. Let ν_2 be the volume form on S^2.

Lemma 4.1. (a) We have $\nu_2 \in E(0, \Delta^{1,1}_{S^2})$ and $\tilde\pi^*\nu_2 \in E(2, \Delta^{1,1}_Z)$.
(b) We have $\Delta^0_Z = -(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2)$ and $2\Delta^{0,0}_Z = \Delta^0_Z + 2\sqrt{-1}\zeta_1$.
(c) We have $\Delta^0_Z\Delta^{0,0}_Z = \Delta^{0,0}_Z\Delta^0_Z$.

Proof. It is immediate from the calculations that we have performed previously that $\delta_Z\tilde\pi^*\nu_2 = 2\zeta^1$. We project on $\Lambda^{0,1}Z$ to see $\delta_{Z,2}\tilde\pi^*\nu_2 = \sqrt{-1}(\zeta^0 - \sqrt{-1}\zeta^1)$ so $\mathrm{d}\delta_{Z,2}\tilde\pi^*\nu_2 = 2\zeta^2 \wedge \zeta^3$. Since this belongs to $\Lambda^{1,1}$, and since $\bar\partial\tilde\pi^*\nu_2 = 0$, we have that $\Delta^{1,1}_Z\tilde\pi^*\nu_2 = \bar\partial\delta_{Z,2}\tilde\pi^*\nu_2 = 2\tilde\pi^*\nu_2$. Let $\star$ be the Hodge operator. We compute

$$\xi_0 = \tfrac{1}{2}(\zeta_0 - \sqrt{-1}\zeta_1) \qquad \xi_1 = \tfrac{1}{2}(\zeta_2 - \sqrt{-1}\zeta_3)$$
$$\xi^0 = \zeta^0 + \sqrt{-1}\zeta^1 \qquad \xi^1 = (\zeta^2 + \sqrt{-1}\zeta^3)$$
$$\partial f = \xi_0 f\cdot\xi^0 + \xi_1 f\cdot\xi^1 \qquad \bar\partial f = \bar\xi_0 f\cdot\bar\xi^0 + \bar\xi_1 f\cdot\bar\xi^1$$
$$\star(\bar\xi^0) = -\tfrac{1}{2}\bar\xi^0\wedge\xi^1\wedge\bar\xi^1 \qquad \star(\bar\xi^1) = \tfrac{1}{2}\bar\xi^1\wedge\xi^0\wedge\bar\xi^0 \qquad \mathrm{d}(\star\bar\xi^0) = \mathrm{d}(\star(\bar\xi^1) = 0.$$

Since $\delta_2 = \delta = -\star\mathrm{d}\star$ on $C^\infty\Lambda^{0,1}$, we have

$$\begin{aligned}\Delta^{0,0}_Z f &= -\star\mathrm{d}\star(\bar\xi_0 f\cdot\bar\xi^0 + \bar\xi_1 f\cdot\xi^1) = \tfrac{1}{2}\star\mathrm{d}(\bar\xi_0 f\cdot\bar\xi^0\wedge\xi^1\wedge\bar\xi^1 + \bar\xi_1 f\cdot\bar\xi^1\wedge\xi^0\wedge\bar\xi^0)\\
&= \tfrac{1}{2}\star(\xi_0\bar\xi_0 f + \xi_1\bar\xi_1 f)\xi^0\wedge\bar\xi^0\wedge\xi^1\wedge\bar\xi^1 = -2(\xi_0\bar\xi_0 + \xi_1\bar\xi_1)f\\
&= -\tfrac{1}{2}(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \sqrt{-1}[\zeta_0, \zeta_1] + \sqrt{-1}[\zeta_2, \zeta_3])\\
&= -\tfrac{1}{2}(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 - 2\sqrt{-1}\zeta_1)\end{aligned}$$
$$\Delta^{0,0}_Z + \bar\Delta^{0,0}_Z = \Delta^0_Z = -(\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2)$$
$$\Delta^{0,0}_Z - \bar\Delta^{0,0}_Z = 2\sqrt{-1}\zeta_1,\ \ 2\Delta^{0,0}_Z = \Delta^0_Z + 2\sqrt{-1}\zeta_1.$$

We show $\Delta^{0,0}_Z$ and Δ^0_Z commute by checking

$$\begin{aligned}-[\zeta_1, \Delta^0_Z] &= \{[\zeta_1, \zeta_2^2] + [\zeta_1, \zeta_3^2]\} = [\zeta_1, \zeta_2]\zeta_2 + \zeta_2[\zeta_1, \zeta_2] + [\zeta_1, \zeta_3]\zeta_3 + \zeta_3[\zeta_1, \zeta_3]\\
&= -2\zeta_3\zeta_2 - 2\zeta_2\zeta_3 + 2\zeta_2\zeta_3 + 2\zeta_3\zeta_2 = 0.\end{aligned}$$

□

Remark 4.2. We can now give the spectral resolution of $\Delta^{0,0}$ on the Hopf manifold; these results also follow from work of Bedford and Suwa [3] who used a different approach. The map $e^{it} : (\lambda, \boldsymbol{x}) \to (e^{it}\lambda, \boldsymbol{x})$ gives a circle action on Z by isometries. We decompose $L^2(Z) = \oplus_j \lambda^j \cdot L^2(S^3)$ under this action. The real and complex Laplacians on Z respect this decomposition so it suffices to study $\Delta^0_{S^3}$ and $\Delta^0_{S^3} + 2\sqrt{-1}\zeta_1$ to determine the spectral resolution. Let $\boldsymbol{x}$ be the standard coordinates on S^3. Let $\mathcal{H}(k)$ be the space of harmonic polynomials of degree k. If $p \in \mathcal{H}(k)$, the restriction of p to the sphere is an eigenfunction of the Laplacian and $\Delta^0_Z p = k(k+2)p$. Furthermore all eigenfunctions of the Laplacian on S^3 arise in this fashion. Thus the decomposition $L^2(Z) = \oplus_{j,k}\lambda^j \cdot \mathcal{H}(k)$ is the spectral resolution of the real Laplacian Δ^0_Z and the associated eigenvalue is $j^2 + k(k+2)$.

The vector field ζ_1 is left quaternion multiplication by $\sqrt{-1}$ and generates an isometric circle action $\phi(t) : z \to (\cos(t) + \sqrt{-1}\sin(t))z$ which commutes with the real Laplacian. We decompose the eigenspace of the Laplacian on S^3 into eigenspaces of this action $\mathcal{H}(k) := \oplus_\ell \mathcal{H}(k, \ell)$ where $\phi(t)^* p = e^{\sqrt{-1}\ell t} p$ or equivalently $\zeta_1 p = \sqrt{-1}\ell p$ for $p \in \mathcal{H}(k, \ell)$. This means that we may decompose $L^2(Z) = \oplus_{j,k,\ell}\lambda^j \cdot \mathcal{H}(k, \ell)$ where the eigenvalue of Δ^0_Z is $j^2 + k(k+2)$ and the eigenvalue of $2\Delta^0_Z$ is $j^2 + k(k+2) - 2\ell$. Let $z^0 = x^0 + \sqrt{-1}x^1$ and $z^1 = x^2 + \sqrt{-1}x^3$. These are not, of course, holomorphic functions on Z. We note $\zeta_1 z^i = \sqrt{-1}z^i$ and $\zeta_1 \bar{z}^0 = -\sqrt{-1}\bar{z}^i$. The $\{z^0, z^1, \bar{z}^0, \bar{z}^1\}$ generate the algebra of all polynomials. Thus we see that we need $-k \leqslant \ell \leqslant k$ when studying $\mathcal{H}(k, \ell)$ so

$$L^2(Z) = \oplus_{j,-k\leqslant\ell\leqslant k}\lambda^j \cdot \mathcal{H}(k, \ell).$$

The space $j = 0$, $\ell = 0$ corresponds to functions which are invariant under both circle actions; such functions are the pull-back of eigenfunctions on the 2-sphere.

5. Conclusion

The Hopf fibration is an example of a non-bijective canonical transformation which arises as the regularization of Kepler motion; the Hopf fibration leads to an inverse harmonic oscillator problem. Pull-back is a transformation that connects operators with different spectra; understanding the pull-back is useful in relating the quantum operators involved.

The real Hopf fibration $S^3 \to S^2$ and the complex Hopf fibration $S^1 \times S^3 \to S^2$ provide examples where the pull-back of a harmonic form from the base of a principal bundle is no longer harmonic but is still an eigenform of the real Laplacian or the complex Laplacian on the total space. We show that eigenvalues cannot change for a principal bundle if the structure group has $H^1(G; \mathbb{C}) = 0$. We give examples of principal bundles over S^3 with structure group G where eigenvalues change if $H^1(G; \mathbb{C}) \neq 0$. Further investigation of this phenomena seems indicated to find other non-trivial examples; the eigenforms whose eigenvalues change seem to be important in understanding the spectral geometry involved.

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References

[1] Asorey M, Carinena J and Ibort L 1983 *J. Math. Phys.* **24** 2745–50
[2] Adams J F 1962 *Ann. Math.* **75** 603–32
[3] Bedford E and Suwa T 1976 *Proc. AMS* **60** 259–64

[4] Bérard Bergery L and Bourguignon J-P 1982 *Illinois J. Math.* **26** 181–200
[5] Boiteux M 1982 *J. Math. Phys.* **23** 1311–14
[6] Bott R and Milnor J 1958 *Bull. AMS* **64** 87–9
[7] Cerdeira H 1985 *J. Phys. A: Math. Gen.* **18** 2719–27
[8] Dehgahni M H and Sobouti Y 1993 *Astron. Astrophys.* **275** 91–5
[9] Eguchi T, Gilkey P B and Hanson A J 1980 *Phys. Rep.* **66** 213–393
[10] Gilkey P B, Leahy J V and Park J H 1996 *Preprint*
[11] Gilkey P B and Park J H *Illinois J. Math.* to appear
[12] Goldberg S I and Ishihara T 1978 *J. Diff. Geom.* **13** 139–44
[13] Gracia-Bondía J M 1984 *Phys. Rev.* A **30** 691–7
[14] Kibler N and Négadi T 1983 *Lett. Nuovo Cimento* **37** 225–8
[15] Kibler N and Négadi T 1984 *Croatia Chem. Acta* **57** 1509–23
[16] Kibler M, Ronveaux A and Négadi T 1986 *J. Math. Phys.* **27** 1541–8
[17] Kibler M and Winternitz P 1988 *J. Phys. A: Math. Gen.* **21** 1787–1803
[18] Mello P A and Moshinsky M 1975 *J. Math. Phys.* **16** 2017–28
[19] Muto Y 1978 *J. Korean Math. Soc.* **15** 39–57
[20] Muto Y 1978 *Kodai Math. J.* **1** 329–38
[21] Park J H 1990 *Bull. Korean Math. Soc.* **27** 39–47
[22] Watson B 1973 *J. Diff. Geom.* **8** 85–94